Appendix H – Estimation of Rasch parameters by iterative proportional fitting

**H.1 Introduction**

Iterative proportional fitting (IPF) provides a simple way to estimate parameters of Rasch models. IPF procedures are sometimes referred to as raking. Rasch described the method as a “svikmølle” (switch mill), the Danish name of the Nine Men’s Morris game.

The IPF procedure can be used to fit an observed multiway table to a new table with different marginal tables in a way that preserves the parameters describing the association between the two variables. We refer to Bishop, Fienberg and Holland (1975) for a comprehensive introduction to iterative proportional fitting during log-linear analysis of multiway tables. The purpose of this appendix is to illustrate the application of the IPF svikmølle during analysis of two-way tables and to show how to use the svikmølle to estimate parameters of Rasch models for dichotomous items.

**H.2 Iterative proportional fitting of two-way tables**

Let X be a two-way (I J) table of cells (xij) with row and column sums  and .

IPF assumes that the xij cells are products of row parameters = (1, …, R), column parameters = (1, …, C), interaction parameters = (11, … , RC), and functions fij(,,),

 (H.1)

We refer to A and B as main effects and say that  describes the interaction structure within the table. The IPF assumes that this structure is independent of the margins of the table in the sense that other tables exist with different main effects but the same interaction structure. The fij functions may be different functions depending on i and j, but many applications of the IPF including applications for log-linear analysis of tables without structural zeros assume that fij(,,) = f(,,). In tables with structural zeros, the IPF assumes that fij(,,) = 0 if xij = 0 is a structural zero.

Let **V** = (V1, … , VR) and **W** = (W1, … , WC) be different sets of row and column sums and assume that we want to find a table defined by

 (H.2)

and

 and  (H.3)

with different main effects,  and , but the same interaction structure as in (H.1).

The IPF is a stepwise procedure that provide solutions to such problems. It starts with an initial set of parameters,  and  and proceeds until an a new sets of main effects that satisfies (H.2) and (H.3) has been found.

During each step, IPF calculates the row and column sums

(H.4)

(H.5)

If the differences between the row and column sums defined by (H.3) and (H.4) and **V** and **W** are small, IPF selects , , and () as the estimates. If not, the IPF updates the,  parameters by multiplication of the current estimates with the ratios between the values of the desired margins sums and the current row and columns sums[[1]](#footnote-2)

 (H.6)

 (H.7)

(H.8)

The reasons for updating the parameters in this simple way is that (H.1) and (H.2) shows that row and column sums are product of on one hand the parameter of interest and on the other hand a complicated function of the complete set of parameters. The suggested updates of the first factors in (H.1) and (H.2) therefore pulls the row and column sums in the correct direction. Deming & Stephan (1940), Fienberg (1970), and Csiszar (1975) are among those who has proven that the intuition actually works.

**H.3. Testing marginal homogeneity**

We illustrate the application of the IPF for two-way tables by a test of marginal homogeneity of two dependent categorical variables.

Table H.1 shows the joint distribution of two polytomous items from a summated scale measuring disinhibited eating among diabetes patients. Response categories are ordinal coded from zero to three.

**Table H1. Joint distribution of DHP36 and DHP39 from the DE scale.**

**2 = 33,7, df = 9, p < 0.0005. = 0.45, p < 0.00**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **DHP36** | | | |  |
| **DHP39** | **0** | **1** | **2** | **3** | **Total** |
| **0** | **21** | **5** | **4** | **4** | **34** |
| **1** | **38** | **24** | **42** | **8** | **112** |
| **2** | **3** | **5** | **8** | **4** | **20** |
| **3** | **3** | **4** | **10** | **9** | **26** |
| **Total** | **65** | **38** | **64** | **25** | **192** |

The items are supposed to depend on the same latent trait. It is therefore no surprise that the items are marginally dependent. However, the hypothesis that might be of interest is whether the *conditional* distribution of the items *given* the latent variable is the same:

H0 : Pr(DHP36=x | θ) = Pr(DHP39=x | θ) for x = 0,1,2,3

H0 is a hypothesis of conditional homogeneity. Compared to this the hypothesis of marginal homogeneity is

HMH: Pr(DHP36=x) = Pr(DHP39=x) for x = 0,1,2,3

Since items are locally dependent it follows that HMH is true if H0 is true and therefore that we have to reject H0 if a test of marginal homogeneity rejects HMH.

To test HMH we need to estimate the expected counts in table H1 under HMH in a way that takes the marginal dependence into account. The canonical parameters describing the association between categorical variables are the so-called local odds-ratios described in Appendix F. Since multiplying rows or columns of contingency tables with arbitrary set of constants does not change the local odds-ratios it follows that the odds-ratios will stay the same during each IPF step. To estimate the expected counts under HMH we therefore fit the H1 table to new a table with row and column sums equal to the averages of the marginal frequencies in H1: 49.5, 75.0, 42.0 and 25.5.

Figure H.1 shows what happens during the first IPF step. Step 0 is the initial observed table with the homogeneous margins added. Following that, each IPF step consists of two substeps. The first fit the columns sums to the homogeneous margins multiplying each cell by the ratio between the homogeneous and current column margin. The second substep count the row sums of the new table and multiply the cell counts by the ratio between the preferred row sums and the row sums of the table.

After the second substep, row sums are as expected, but column sums do not fit.  is the largest difference between estimated and preferred margins. Since this is too large, the IPF proceeds with the additional step as shown in Figure H.2. At the end of this step, fit is much better but not good enough,. Three more steps needed before the solution is accepted. Figure H.3 shows the result.

***Step no. 0***

**1 2 3 4 total V**

**1 21.00 5.00 4.00 4.00 34.00 49.50**

**2 38.00 24.00 42.00 8.00 112.00 75.00**

**3 3.00 5.00 8.00 4.00 20.00 42.00**

**4 3.00 4.00 10.00 9.00 26.00 25.50**

**Total 65.00 38.00 64.00 25.00**

**W 49.50 75.00 42.00 25.50**

***Step no. 1 a***

**1 2 3 4 total V**

**1 15.99 9.87 2.63 4.08 32.57 49.50**

**2 28.94 47.37 27.56 8.16 112.03 75.00**

**3 2.28 9.87 5.25 4.08 21.48 42.00**

**4 2.28 7.89 6.56 9.18 25.92 25.50**

**Total 49.50 75.00 42.00 25.50**

**W 49.50 75.00 42.00 25.50**

***Step no. 1 b***

**1 2 3 4 total V**

**1 24.31 15.00 3.99 6.20 49.50 49.50**

**2 19.37 31.71 18.45 5.46 75.00 75.00**

**3 4.47 19.29 10.26 7.98 42.00 42.00**

**4 2.25 7.77 6.46 9.03 25.50 25.50**

**Total 50.40 73.77 39.16 28.67**

**W 49.50 75.00 42.00 25.50**

**Delta = 3.17159**

**Figure H1. The first IPF step attempting to fit Table H.1 to the homogeneous margins.**

***Step no. 2 a***

**1 2 3 4 total V**

**1 23.88 15.25 4.28 5.52 48.92 49.50**

**2 19.03 32.24 19.79 4.86 75.92 75.00**

**3 4.39 19.61 11.01 7.09 42.10 42.00**

**4 2.21 7.90 6.92 8.03 25.06 25.50**

**Total 49.50 75.00 42.00 25.50**

**W 49.50 75.00 42.00 25.50**

***Step no. 2 b***

**1 2 3 4 total V**

**1 24.16 15.43 4.33 5.58 49.50 49.50**

**2 18.80 31.85 19.55 4.80 75.00 75.00**

**3 4.38 19.57 10.98 7.08 42.00 42.00**

**4 2.25 8.03 7.05 8.17 25.50 25.50**

**Total 49.58 74.88 41.91 25.63**

**W 49.50 75.00 42.00 25.50**

**Delta = 0.13069**

**Figure H.2. The second IPF step attempting to fit Table H.1 to homogeneous margins.**

The example illustrates that IPF for contingency tables is extremely efficient. In this case, where we only need to calculate a test comparing Table H.1 to the final table in Figure H.3 there is no reason to look at the main effects and the interaction parameters. The likelihood ratio test comparing the two tables rejects the hypothesis of marginal homogeneity (LR = 71.3, df = 3, p < 0.0005).

***Step no. 5 a***

**1 2 3 4 total V**

**1 24.13 15.47 4.34 5.56 49.50 49.50**

**2 18.76 31.89 19.59 4.77 75.00 75.00**

**3 4.37 19.59 11.00 7.04 42.00 42.00**

**4 2.24 8.05 7.07 8.14 25.50 25.50**

**Total 49.50 75.00 42.00 25.50**

**W 49.50 75.00 42.00 25.50**

***Step no. 5 b***

**1 2 3 4 total V**

**1 24.13 15.47 4.34 5.56 49.50 49.50**

**2 18.76 31.89 19.59 4.77 75.00 75.00**

**3 4.37 19.59 11.00 7.04 42.00 42.00**

**4 2.24 8.05 7.07 8.14 25.50 25.50**

**Total 49.50 75.00 42.00 25.50**

**W 49.50 75.00 42.00 25.50**

**Delta = 0.00009**

**Figure H.3. The final IPF step attempting to fit Table H.1 to homogeneous margins.**

**4 Joint estimates of parameters in Rasch models**

Estimating parameters in Rasch models is a larger challenge for IPF than fitting simple two- and multiway-tables. Fortunately, the time needed is rarely a practical problem except for very complicated graphical Rasch models. This and the following sections illustrate IPF for Rasch’s model for dichotomous items.

*Joint estimation* of item and person parameters estimate the item and person parameters simultaneously. This may not be what Rasch preferred, but it is to see what happens if the IPF is used and it will provide examples where we can compare joint estimates to the conditional estimates that Rasch preferred.

***Example 1 – 20 persons and four items***

The first example consists of simulated responses from twenty persons to four items. Eight persons had extreme scores. Figure H.4 shows the responses to the items for the remaining twelve persons. We use these responses to estimate item and person parameters.

**Items**

**pers. 1 2 3 4 R**

**1 0 1 1 1 3**

**2 0 1 1 0 2**

**3 0 1 0 0 1**

**4 1 0 0 0 1**

**5 1 0 0 0 1**

**6 1 0 1 0 2**

**7 0 0 1 0 1**

**8 1 0 0 0 1**

**9 0 1 1 0 2**

**10 0 0 1 0 1**

**11 0 1 0 0 1**

**12 1 1 1 0 3**

**C 5 6 7 1**

**Figure H.4 Responses to four dichotomous items. R is the person score and C is the item margin.**

The multiplicative version of the Rasch model defines probabilities in a way that fit the multiplicative structure defined required by iterative proportional fitting.

 (H.7)

In (H.7), the and parameters correspond to and parameters of Formula (H.1), the interaction parameters are equal to one and fij(,,) is equal to 1/(1+ij).

To calculate the joint estimates we search for person and item parameters, where the row and column margins of the table of estimates probabilities are equal to the person and item scores in Figure H.4,

 and  (H.8)

The IPF needs 53 steps to find a set of parameters satisfying (H.8)

In Figure H.5, Initial item and person parameters are equal to 1, all probabilities are equal to 0.5 and the margins of the table of probabilities differ from the observed person and item scores.

During each step, the IPF redefines person and item parameters as described in Formulas (H.5) and (H.6) and adjusts item parameters and person parameters so that the product of item parameters is equal to 1.0. Figure H.5 show the results after the first two steps and Figure H.6 shows the result after the fifth and the fifty-third step.

Figure H.6 shows that all persons with the same score get the same estimate of the person parameters. This is hardly a surprise because of the sufficiency of the person score, but useful to be reminded about, because it suggest that IPF estimation of person parameters could have been a little easier than in this example. We return to this in the next example. For now, Tables H.2 and H.3 display the joint estimates of the parameters. Because of the small sample size, it is no surprise that there is error associated with the estimates of the item parameters. Persons were randomly selected from a standardized normal distribution of ln(Xi). The true values of the person parameter were not recorded.

**Table H.2 Joint estimates of item parameters**

**Item Delta -ln(delta) Beta**

**-----------------------------**

**1 1.403 -0.338 -0.25**

**2 2.155 -0.768 -0.75**

**3 3.261 -1.182 -1.25**

**4 0.101 2.288 2.25**

**Table H.3 Joint estimates of person parameters**

**score Xi ln(Xi)**

**-----------------------**

**1 0.221 -1.509**

**2 0.844 -0.169**

**3 4.321 1.463**

***Initialization***

**person 1 2 3 4 total R Xi**

**1 0.500 0.500 0.500 0.500 2.00 3 1.000**

**2 0.500 0.500 0.500 0.500 2.00 2 1.000**

**3 0.500 0.500 0.500 0.500 2.00 1 1.000**

**4 0.500 0.500 0.500 0.500 2.00 1 1.000**

**5 0.500 0.500 0.500 0.500 2.00 1 1.000**

**6 0.500 0.500 0.500 0.500 2.00 2 1.000**

**7 0.500 0.500 0.500 0.500 2.00 1 1.000**

**8 0.500 0.500 0.500 0.500 2.00 1 1.000**

**9 0.500 0.500 0.500 0.500 2.00 2 1.000**

**10 0.500 0.500 0.500 0.500 2.00 1 1.000**

**11 0.500 0.500 0.500 0.500 2.00 1 1.000**

**12 0.500 0.500 0.500 0.500 2.00 3 1.000**

**total 6.0 6.0 6.0 6.0 12**

**C 5 6 7 1**

**Delta 1.000 1.000 1.000 1.000**

***Step no. 1***

**person 1 2 3 4 total R Xi**

**1 0.598 0.641 0.676 0.230 2.15 3 1.134**

**2 0.498 0.544 0.582 0.166 1.79 2 0.756**

**3 0.332 0.373 0.410 0.090 1.21 1 0.378**

**4 0.332 0.373 0.410 0.090 1.21 1 0.378**

**5 0.332 0.373 0.410 0.090 1.21 1 0.378**

**6 0.498 0.544 0.582 0.166 1.79 2 0.756**

**7 0.332 0.373 0.410 0.090 1.21 1 0.378**

**8 0.332 0.373 0.410 0.090 1.21 1 0.378**

**9 0.498 0.544 0.582 0.166 1.79 2 0.756**

**10 0.332 0.373 0.410 0.090 1.21 1 0.378**

**11 0.332 0.373 0.410 0.090 1.21 1 0.378**

**12 0.598 0.641 0.676 0.230 2.15 3 1.134**

**total 5.0 5.5 6.0 1.6 12**

**C 5 6 7 1**

**Delta 1.313 1.576 1.839 0.263**

***Step no. 2***

**person 1 2 3 4 total R Xi**

**1 0.677 0.733 0.777 0.200 2.39 3 1.496**

**2 0.528 0.594 0.649 0.118 1.89 2 0.797**

**3 0.293 0.352 0.407 0.047 1.10 1 0.296**

**4 0.293 0.352 0.407 0.047 1.10 1 0.296**

**5 0.293 0.352 0.407 0.047 1.10 1 0.296**

**6 0.528 0.594 0.649 0.118 1.89 2 0.797**

**7 0.293 0.352 0.407 0.047 1.10 1 0.296**

**8 0.293 0.352 0.407 0.047 1.10 1 0.296**

**9 0.528 0.594 0.649 0.118 1.89 2 0.797**

**10 0.293 0.352 0.407 0.047 1.10 1 0.296**

**11 0.293 0.352 0.407 0.047 1.10 1 0.296**

**12 0.677 0.733 0.777 0.200 2.39 3 1.496**

**total 5.0 5.7 6.4 1.1 12**

**C 5 6 7 1**

**Delta 1.401 1.838 2.323 0.167**

**Figure H.5. Initialization and steps 1 and 2. Delta and Xi are item and person parameters and C and R are item and person scores.**

***Step no. 5***

**Fitted ARI table**

**person 1 2 3 4 total R Xi**

**1 0.778 0.836 0.878 0.245 2.74 3 2.545**

**2 0.550 0.640 0.715 0.101 2.01 2 0.886**

**3 0.255 0.332 0.413 0.031 1.03 1 0.248**

**4 0.255 0.332 0.413 0.031 1.03 1 0.248**

**5 0.255 0.332 0.413 0.031 1.03 1 0.248**

**6 0.550 0.640 0.715 0.101 2.01 2 0.886**

**7 0.255 0.332 0.413 0.031 1.03 1 0.248**

**8 0.255 0.332 0.413 0.031 1.03 1 0.248**

**9 0.550 0.640 0.715 0.101 2.01 2 0.886**

**10 0.255 0.332 0.413 0.031 1.03 1 0.248**

**11 0.255 0.332 0.413 0.031 1.03 1 0.248**

**12 0.778 0.836 0.878 0.245 2.74 3 2.545**

**total 5.0 5.9 6.8 1.0 12**

**C 5 6 7 1**

**Delta 1.380 2.008 2.836 0.127**

***Step no. 53***

**Fitted ARI table**

**person 1 2 3 4 total R Xi**

**1 0.858 0.903 0.934 0.305 3.00 3 4.321**

**2 0.542 0.645 0.734 0.079 2.00 2 0.844**

**3 0.237 0.323 0.419 0.022 1.00 1 0.221**

**4 0.237 0.323 0.419 0.022 1.00 1 0.221**

**5 0.237 0.323 0.419 0.022 1.00 1 0.221**

**6 0.542 0.645 0.734 0.079 2.00 2 0.844**

**7 0.237 0.323 0.419 0.022 1.00 1 0.221**

**8 0.237 0.323 0.419 0.022 1.00 1 0.221**

**9 0.542 0.645 0.734 0.079 2.00 2 0.844**

**10 0.237 0.323 0.419 0.022 1.00 1 0.221**

**11 0.237 0.323 0.419 0.022 1.00 1 0.221**

**12 0.858 0.903 0.934 0.305 3.00 3 4.321**

**total 5.0 6.0 7.0 1.0 12**

**C 5 6 7 1**

**Delta 1.403 2.155 3.261 0.101**

**Figure H.6. Step 5 and estimates of probabilities and parameters at step 53.**

***Example 2***

The next example is more realistic using the data from Chapter 2 where seven items fitted the Rasch model for dichotomous items. 517 persons had scores between one and six. Instead of using IPF to fit a 517 7 table to the raw scores, we collect persons with the same score in groups and define the expected counts in the cells of a 6 7 table defined by

 (H.9)

Table 4 shows the table counting the number of positive responses to items in different score croups. In this table, the observed row sums are equal to Vr = rnr and W are the item margins Since 110 person have a person score equal to 2, the total number of positive responses in this score group has to be equal to 220.

**Table 4. Frequency of positive responses to seven items. n is the number of persons with a given score, V and W are the row sum and column sums.**

**score**

**r 1 2 3 4 5 6 7 n V**

**1 55 1 17 6 16 1 3 99 99**

**2 78 10 60 9 42 5 16 110 220**

**3 113 16 98 20 77 9 42 125 375**

**4 86 31 80 27 75 20 49 92 368**

**5 58 33 53 26 52 20 48 58 290**

**6 32 27 31 28 32 18 30 33 198**

**W 422 118 339 116 294 73 188 517**

Because of the multiplicative structure of (H.9), IPF provides the joint maximum likelihood estimates of person and item parameter, exactly as in Example 1. In this case, IPF used 114 steps before the convergence was accepted. Figures H.7 and H.8 and Tables H.4 and H.5 show the results. Figure H8 shows the expected counts of positive responses for different items in different score groups. Comparison of these results with the observed count in Table H.4 can be used to test the fit of the Rasch model to data.

***Step no. 1***

**score 1 2 3 4 5 6 7 n total V Xi**

**1 22.0 22.0 22.0 22.0 22.0 22.0 22.0 99 154.00 99 0.238**

**2 40.0 40.0 40.0 40.0 40.0 40.0 40.0 110 280.00 220 0.477**

**3 57.7 57.7 57.7 57.7 57.7 57.7 57.7 125 403.85 375 0.715**

**4 49.1 49.1 49.1 49.1 49.1 49.1 49.1 92 343.47 368 0.954**

**5 34.1 34.1 34.1 34.1 34.1 34.1 34.1 58 238.82 290 1.192**

**6 20.8 20.8 20.8 20.8 20.8 20.8 20.8 33 145.89 198 1.430**

**total 223.7 223.7 223.7 223.7 223.7 223.7 223.7 517**

**W 422 118 339 116 294 73 188**

**Delta 2.261 0.632 1.816 0.621 1.575 0.391 1.007**

***Step no. 2***

**score 1 2 3 4 5 6 7 n total V Xi**

**1 26.6 9.2 22.6 9.1 20.2 5.9 13.9 99 107.64 99 0.154**

**2 52.6 22.4 46.6 22.1 42.9 15.0 31.9 110 233.55 220 0.384**

**3 77.5 39.2 70.9 38.7 66.5 27.5 52.7 125 373.16 375 0.685**

**4 65.8 38.0 61.5 37.6 58.6 27.9 48.6 92 338.10 368 1.055**

**5 45.3 28.9 43.0 28.7 41.3 22.1 35.6 58 244.89 290 1.493**

**6 27.3 18.8 26.2 18.7 25.4 14.9 22.4 33 153.70 198 1.997**

**total 295.2 156.6 270.9 155.0 254.9 113.4 205.1 517**

**W 422 118 339 116 294 73 188**

**Delta 3.410 0.502 2.398 0.491 1.917 0.266 0.974**

**Figure H.7 Steps 1 and of the search for joint estimates. In these tables, n is the number of persons with the given score, V and W are the preferred marginal totals and Delta and Xi are parameters.**

***Step no. 10***

**score 1 2 3 4 5 6 7 n total V Xi**

**1 43.0 3.0 25.0 2.9 17.6 1.5 6.7 99 99.76 99 0.093**

**2 77.4 9.7 56.1 9.4 44.0 4.9 20.2 110 221.64 220 0.286**

**3 106.3 23.5 89.3 23.0 77.0 12.7 43.8 125 375.52 375 0.688**

**4 85.2 31.0 77.8 30.4 71.6 18.3 49.9 92 364.28 368 1.511**

**5 55.9 30.1 53.4 29.7 51.1 20.0 41.5 58 281.58 290 3.198**

**6 32.4 22.9 31.7 22.7 31.0 17.4 27.8 33 185.96 198 6.767**

**total 400.2 120.1 333.3 118.1 292.4 74.8 189.9 517**

**W 422 118 339 116 294 73 188**

**Delta 8.731 0.330 3.696 0.321 2.342 0.160 0.777**

***Step no. 10***

**score 1 2 3 4 5 6 7 n total R Xi**

**1 48.8 2.2 23.8 2.2 15.7 1.0 5.4 99 99.00 99 0.078**

**2 84.1 7.8 56.5 7.6 42.5 3.8 17.7 110 220.00 220 0.262**

**3 111.8 20.9 91.8 20.4 77.8 10.6 41.7 125 375.00 375 0.684**

**4 87.7 30.1 80.0 29.5 73.5 16.8 50.4 92 368.00 368 1.653**

**5 56.9 31.4 54.6 30.9 52.6 20.4 43.3 58 290.00 290 4.010**

**6 32.8 25.6 32.3 25.5 31.9 20.3 29.6 33 198.00 198 11.852**

**total 422.0 118.0 339.0 116.0 294.0 73.0 188.0 517**

**C 422 118 339 116 294 73 188**

**Delta 12.400 0.293 4.037 0.285 2.409 0.135 0.732**

**Figure H.8 Estimates of joint parameters at step 10 and after 114 steps.**

**Table H.4 Joint estimates of item parameters. The locations of the items, -ln(i), according to the IRT version of the model are also included.**

**Item Delta -ln(delta)**

**-----------------------**

**1 12.400 -2.518**

**2 0.293 1.226**

**3 4.037 -1.396**

**4 0.285 1.256**

**5 2.409 -0.879**

**6 0.135 1.999**

**7 0.732 0.312**

**Table H.5 Joint estimates of item parameters. The locations of the person, ln(r), are also included.**

**score Xi ln(Xi)**

**-----------------------**

**1 0.078 -2.548**

**2 0.262 -1.341**

**3 0.684 -0.380**

**4 1.653 0.503**

**5 4.010 1.389**

**6 11.852 2.473**

Dividing the expected count  in Figure H.8 with the number of persons nr that responded to item j provides an estimate of the conditional probability of a positive response to the item given the vectors of the person scores and the item margins. That is, the conditional probabilities in the frame of inference for tests of the Rasch model proposed by Rasch (1961). Table H.6 shows these probabilities and Table H.7 shows the estimates of the same probabilities by the Markov Chain Monte Carlo procedure proposed by Besag & Clifford (1989).

The result is remarkable because it shows that the probability of a response to an item given the vector of person scores R and item margins C is equal to the Rasch models probabilities with the joint estimates inserted instead of the true parameters.

 (H.10)

**Table H.6 Conditional probabilities of responses to items given person scores and item margins.**

**Item**

**score 1 2 3 4 5 6 7**

**1 0.492 0.022 0.240 0.022 0.159 0.010 0.054**

**2 0.764 0.071 0.514 0.069 0.387 0.034 0.161**

**3 0.895 0.167 0.734 0.163 0.622 0.085 0.334**

**4 0.953 0.327 0.870 0.320 0.799 0.183 0.548**

**5 0.980 0.541 0.942 0.533 0.906 0.352 0.746**

**6 0.993 0.777 0.980 0.771 0.966 0.616 0.897**

**Table H.7 MCMC estimates of conditional probabilities of responses to items given person scores and item margins based on a random sample of 20,000 tables**

**Item**

**Score 1 2 3 4 5 6 7**

**1 0.522 0.025 0.220 0.024 0.143 0.013 0.053**

**2 0.763 0.073 0.523 0.072 0.376 0.039 0.154**

**3 0.884 0.163 0.746 0.159 0.639 0.088 0.321**

**4 0.943 0.321 0.869 0.315 0.807 0.183 0.563**

**5 0.973 0.545 0.937 0.537 0.905 0.344 0.760**

**6 0.990 0.789 0.976 0.782 0.964 0.596 0.903**

The estimate (H.10) is an unbiased estimate of the conditional probability of a person’s response given the vectors of person scores and item margins, however this does not imply that the estimate of the joint distribution of complete set of item responses is a Rasch model defined by the joint estimates. Item responses in the conditional inference frame defined by R and C are stochastically dependent. Formula (H.10) only applies to a single response.

In addition to this, (H.10) is not an unbiased estimate of the unconditional probabilities of the Rasch model. To calculate the unconditional expected outcome of the estimate of the probability defined by joint estimates we would have to calculate joint estimates and (H.10) for each possible outcomes of (R,C) and calculate the average of (H.10) weighted by Pr(R,C). Using (H.10) during joint inference will result in biased estimates of residuals comparing observed to expected outcomes on items.

**H.5 Conditional estimates of item parameters**

The previous section showed that joint estimates of person parameters can be estimated in score groups, but this does not imply that the parameters are conditional estimates. For that too be true, we should estimate person parameters in the conditional distribution of item responses given the sufficient item margins and item parameters in the conditional distribution given person scores. We return to the estimates of the person parameters in the next section after we have described how to use IPF to provide conditional estimates of item parameters.

The starting point is again Table 4 counting the number of positive item responses in different score groups, where we replace the expected counts of formula (H.9) with expected counts defined by the conditional probabilities of item responses given the person score.

 (H.11)

In (H.11), g(r,) is the symmetrical polynomial of order r defined by the estimated item parameters and g(r-1, \i) is the symmetrical polynomial of order r-1 defined by the set of item parameters without the i’th item. Since (H.11) does not depend on the person parameter and because the sum of the conditional probabilities across all items is equal to 1, the row sums will always be equal to the number of persons multiplied by the row number which is equal to the raw score, eri =rnr for which reason H.11 reduces to

 (H.12)

For this reason, the IPF procedure simplifies because we only have to adjust the column sums to fit the observed item margins. To speed things up, we define initial item parameter values by the relative frequencies of the item margins (standardized so that the product is equal to one). For this reason estimation only need 44 steps. Figures H.9 and H.10 show test output and Table H.8 6 shows the estimates.

***Step no. 1***

**score 1 2 3 4 5 6 7 n**

**1 55.0 1.0 17.0 6.0 16.0 1.0 3.0 99**

**2 86.1 3.5 49.6 19.9 47.2 3.5 10.2 110**

**3 112.1 9.1 87.5 46.1 85.6 9.1 25.5 125**

**4 87.7 14.3 78.6 58.4 77.8 14.3 36.8 92**

**5 56.9 18.5 54.3 48.1 54.1 18.5 39.5 58**

**6 32.8 20.5 32.3 30.9 32.2 20.5 28.8 33**

**total 403.8 112.9 324.4 111.0 281.4 69.9 179.9**

**W 422.0 118.0 339.0 116.0 294.0 73.0 188.0**

**delta 8.647 0.283 2.894 0.533 2.410 0.175 0.628**

***Step no. 2***

**score 1 2 3 4 5 6 7 n**

**1 55.0 1.8 18.4 3.4 15.3 1.1 4.0 99**

**2 86.3 6.2 52.9 11.5 45.7 3.9 13.5 110**

**3 112.1 16.0 90.3 28.7 84.6 10.1 33.1 125**

**4 87.4 24.2 78.8 40.3 76.3 15.8 45.3 92**

**5 56.6 27.0 54.0 38.9 53.2 18.8 41.4 58**

**6 32.7 24.4 32.2 28.4 32.0 19.1 29.1 33**

**total 418.5 117.0 336.2 115.0 291.5 72.4 186.4**

**W 422.0 118.0 339.0 116.0 294.0 73.0 188.0**

**delta 8.414 0.332 2.980 0.406 2.287 0.184 0.704**

**Step no. 10**

**score 1 2 3 4 5 6 7 n**

**1 52.3 2.4 21.4 2.4 14.1 1.3 5.2 99**

**2 84.3 8.0 57.3 7.8 41.4 4.3 16.9 110**

**3 110.8 20.3 93.0 19.8 80.0 11.0 40.1 125**

**4 86.9 29.5 79.9 28.9 74.3 16.8 51.8 92**

**5 56.5 31.6 54.3 31.1 52.5 19.9 44.0 58**

**6 32.7 26.0 32.2 25.8 31.8 19.7 29.8 33**

**total 421.8 117.9 338.8 115.9 293.9 73.0 187.9**

**W 422.0 118.0 339.0 116.0 294.0 73.0 188.0**

**delta 7.804 0.361 3.211 0.352 2.115 0.189 0.782**

**Figure H.9 Steps 1, 2 and 10 of the search for conditional item estimates. In these tables, n is the number of persons with the given score and W is the preferred marginal totals.**

***Step no. 44***

**Expected ARI**

**score 1 2 3 4 5 6 7 n**

**1 51.7 2.4 21.7 2.4 14.2 1.3 5.3 99**

**2 83.9 8.1 57.6 7.9 41.4 4.3 17.0 110**

**3 110.5 20.3 93.2 19.9 79.9 11.0 40.2 125**

**4 86.7 29.5 80.0 28.9 74.2 16.8 51.8 92**

**5 56.4 31.6 54.3 31.1 52.5 19.9 44.1 58**

**6 32.7 26.0 32.2 25.8 31.8 19.7 29.8 33**

**total 422.0 118.0 339.0 116.0 294.0 73.0 188.0**

**C 422.0 118.0 339.0 116.0 294.0 73.0 188.0**

**delta 7.694 0.362 3.230 0.353 2.115 0.190 0.784**

**Figure H.10 Estimates of conditional estimates of item parameters after 44 steps.**

**Table H.8 Conditional estimates of item parameters. The locations of the person, ln(i), are also included.**

**Item delta -ln(delta)**

**------------------------**

**1 7.694 -2.040**

**2 0.362 1.015**

**3 3.230 -1.172**

**4 0.353 1.041**

**5 2.115 -0.749**

**6 0.190 1.663**

**7 0.784 0.243**

**6. Conditional estimation of person parameters**

Since the Rasch model for dichotomous items is symmetrical in items and persons, it is obvious that the IPF algorithm should function in exactly the same way for person parameters as for item parameters except that the expected cell count has to be defined by the conditional distribution of item scores given the *item* margins.

 (H.13)

In (H.12), is the vector of person estimates for all 517 persons with scores between 1 and 6, is the vector where one person parameter has been removed, and wi is the total number of positive responses to the i’th item.

Joint and conditional estimation of item parameters selects the origin of the logit scale so that the sum of IRT parameters are equal to zero. This option does not exist during conditional estimation of person parameters where item parameters are ignored. Instead, we estimate referenced person estimates, where we assume that the estimate of the multiplicative person parameter corresponding to the mid-score (in this case 3) is equal to 1.

We select initial values defined by the pairwise person estimates suggested by Kreiner (2012) and use the IPF algorithm to fit the expected row scores of Table H.1 to the observed row scores. Figure H.11 shows the initial values and the first couple of steps and table H.9 shows the conditional estimates of person parameters after 61 steps.

Two issues has to be addressed.

The first is that the origin of logit scale of the referenced person parameters differ from the origin of the conditional item estimates. Therefore, we cannot use the difference between the person and item locations to calculate the probabilities of positive responses to items. If we do that, and calculate the expected number of positive responses for all persons and all items, the result will differ from the observed number. For this reason, we use a simplified IPF algorithm to calibrate the estimates of the person parameters by adding a constant to all the estimates so that the expected number of positive responses is equal to the observed number.

The second is that conditional maximum likelihood estimates of person parameters do not provide estimates for extreme and unobserved scores. In this case, there are no unobserved scores, but estimates for extreme scores are missing. In lack of better proposals, we use the same estimates that we use for maximum likelihood estimates where expected true scores are equal to 0.25 and 6.75 respectively. Table H.10 shows the results together with the ML estimates of the person parameters.

**Initial referenced parameters**

**and score distribution**

**Score Parameters n**

**--------------------------**

**1 0.141 -1.961 99**

**2 0.424 -0.858 110**

**3 1.000 0.000 125**

**4 2.204 0.790 92**

**5 4.989 1.607 58**

**6 14.039 2.642 33**

**--------------------------**

***Step no = 1***

**Score W Fit Xi**

**--------------------------**

**1 99 109.51 0.127**

**2 110 114.09 0.409**

**3 125 124.95 1.000**

**4 92 90.18 2.249**

**5 58 56.52 5.120**

**6 33 32.36 14.314**

***Step no = 2***

**Score W Fit Xi**

**--------------------------**

**1 99 104.41 0.121**

**2 110 113.20 0.397**

**3 125 125.56 0.996**

**4 92 90.88 2.276**

**5 58 56.86 5.220**

**6 33 32.44 14.555**

**Step no = 10**

**Score W Fit Xi**

**--------------------------**

**1 99 99.26 0.115**

**2 110 110.29 0.383**

**3 125 125.27 0.998**

**4 92 92.04 2.395**

**5 58 57.87 5.729**

**6 33 32.81 16.317**

**Figure H.11 Initial parameters and steps 1, 2 and 10 of the search for conditional person estimates estimates.**

**Table H.9 Conditional estimates of person parameters. The locations of the person, ln(r), are also included.**

**Score Parameters n**

**--------------------------**

**1 0.115 -2.163 99**

**2 0.383 -0.960 110**

**3 1.000 0.000 125**

**4 2.412 0.880 92**

**5 5.834 1.764 58**

**6 17.179 2.844 33**

**--------------------------**

**Table H.10 Calibrated conditional estimates of person parameters.**

**referenced calibrated**

**score n CML CML ML**

**-------------------------------------------**

**0 ---- -3.988 -3.988**

**1 99 -2.163 -2.524 -2.322**

**2 110 -0.960 -1.321 -1.219**

**3 125 0.000 -0.361 -0.361**

**4 92 0.880 0.519 0.429**

**5 58 1.764 1.403 1.246**

**6 33 2.844 2.483 2.281**

**7 ---- 3.887 3.887**

**-------------------------------------------**

Calibrated CML estimates = ML estimates when CML estimates are not available

CML estimates for extreme scores are pseudo ML estimates with expected scores = 0.25 and 6.75

**7 Final comments**

The IPF algorithm functions for conditional estimation of item parameters for all the models described in this book, including the models for polytomous items and graphical and log-linear Rasch models. There is no reason why the methods should not work for conditional estimation of person parameters in such models, but to our knowledge, it has not been implemented in any programs for analysis by Rasch models. Appendix L compares different estimates of person parameters of person parameters in the model for dichotomous items.

**References**

Besag, J. & Clifford. P. (1989). Generalized Monte Carlo Significance Tests. *Biometrika,* 76, 633-642

Bishop, Y.M.M., Fienberg, S.E. & Holland, P.W. (1975) *Discrete Multivariate Analysis: Theory and Practice*. Cambridge: MIT Press.

Csiszar, I. (1975) l-divergence of probability distributions and minimization problems. *Annals of Probability*, **3**, 146-158

Deming, W.E. & Stephan, F. F. (1940) On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. *Annals of Mathematical Statistics*, **11**, 427-444

Fienberg, S.E. (1970) An iterative procedure for estimation in contingency tables. *Annals of Mathematical Statistics*, **41**, 907-917

Kreiner, S. (2012)Conditional pairwise person parameter estimates in Rasch models. Journal of Applied Measurement, 13, 314-320

1. To reduce the number of steps needed it will be convenient to recalculate (H.3) and (H.5) with instead of before (H.7) or (H.3) and (H.4) before (H.6).

   

   [↑](#footnote-ref-2)